

Local finite dimensional Gorenstein k -algebras having Hilbert function $(1, 5, 5, 1)$ are smoothable

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Abstract

Let k be an algebraically closed field of characteristic 0. The question of irreducibility of the punctual Hilbert scheme $\mathcal{Hilb}_d \mathbb{P}_k^n$ and its Gorenstein locus for various d was studied in [CEVV8, CN9, CN10, CN11]. In this short paper we prove that the subschemes corresponding to the Gorenstein algebras having Hilbert function $(1, 5, 5, 1)$ are smoothable i.e. lie in the closure of the locus of smooth subschemes. Among the Gorenstein algebras of length 12 the smoothability of algebras having such Hilbert function seems to be the most inapproachable using non-direct tools e.g. structural theorems.

For a local and finite dimensional (abbreviated f.d.) k -algebra (A, \mathfrak{m}, k) we define its Hilbert function h_A as the Hilbert function of $\text{gr}_{\mathfrak{m}}(A)$, then $h_A(n) = 0$ for $n \gg 0$ and the function is commonly written as the vector of its non-zero values. The *socle degree* of the algebra A is defined by $d := \max\{n : h_A(n) \neq 0\}$. The algebra A is *Gorenstein* iff it is injective as an A -module. Although Gorenstein algebras need not be graded in this paper we encounter only graded ones.

The Hilbert scheme $\mathcal{Hilb}_d \mathbb{P}^n$ parametrizing subschemes of length d has an irreducible component $\mathcal{Hilb}_d^{\circ} \mathbb{P}^n$ being the closure of smooth subschemes, hence the question of irreducibility is the question of equality. A scheme $R = \text{Spec } A \subseteq \mathbb{P}^n$ of length d is called *smoothable* if $[R] \in \mathcal{Hilb}_d^{\circ} \mathbb{P}^n$. This is equivalent to the existence of an abstract smoothing of the algebra A i.e. a flat family of k -algebras with the general fiber smooth and the special fiber isomorphic to A (see [CEVV8] for details).

Every algebra having Hilbert function $h_A = (1, 5, 5, 1)$ can be embedded into $\mathbb{A}_k^5 \hookrightarrow \mathbb{P}_k^5$, thus we fix a 5-dimensional k -linear space with basis

$$V := \bigoplus_{i=1}^5 kx_i \text{ and a dual space with a dual basis } V^* := \bigoplus_{i=1}^5 ka_i, \text{ so that } a_i(x_j) = \delta_{i,j}.$$

Finally we take $W := V^* \oplus kz$ obtaining an inclusion $V^* \subseteq \mathbb{P}W$.

The study of f.d. Gorenstein algebras is closely related to the concept of inverse systems (due to Macaulay) and apolar ideals briefly explained below. The natural pairing $V \otimes V^* \rightarrow k$ may be extended to

$$\lrcorner : S^{\bullet}V^* \otimes S^{\bullet}V \rightarrow S^{\bullet}V.$$

by viewing a_i as a partial derivation operator $\partial/\partial x_i$ acting on $S^{\bullet}V$. Fixing a element $F \in S^{\bullet}V$ (not necessarily homogeneous) of total degree d and restricting \lrcorner to $S^{\bullet}V^* \otimes kF$ we obtain a linear map $S^{\bullet}V^* \rightarrow S^{\bullet}V$. Its kernel I_F is an ideal of $S^{\bullet}V^*$ called the *apolar ideal* of the form F and the residue algebra $S^{\bullet}V^*/I_F$ is called the *apolar algebra* of F . It is a local f.d. Gorenstein k algebra with socle degree d and all such algebras arise this way (see [Eis, Chap. 21] for details). If F was homogeneous with respect to the total degree then I_F is homogeneous and the apolar algebra is graded.

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1 Smoothability

The main result of the paper is

Theorem 1. *Local finite dimensional Gorenstein k -algebras having Hilbert function $(1, 5, 5, 1)$ are smoothable.*

The proof is presented at the end of this section. The same result has been independently obtained by Cristina Bertone, Francesca Cioffi and Margherita Roggero by different means in [BCR].

In the series of papers [CN9, CN10, CN11] Gianfranco Casnati and Roberto Notari proved that the Gorenstein locus of $\text{Hilb}_d \mathbb{P}^n$ is irreducible for $d \leq 11$, $n \geq 1$. The same authors communicated to us that using Theorem 1 they proved

Theorem 2 (G. Casnati, R. Notari — in preparation). *For $d \leq 12$ and $n \geq 1$ the locus of $\text{Hilb}_d \mathbb{P}^n$ consisting of Gorenstein schemes having length d is irreducible.*

The zero-dimensional Gorenstein schemes and their smoothability is also investigated in relation with the study of secant varieties to Veronese reembeddings. As a consequence of the Theorem 2 we obtain

Theorem 3. *Let r, n, d, i be integers and let $W \simeq \mathbb{C}^{n+1}$ be a vector space. Let $\sigma_r(v_d(\mathbb{P}W))$ be the r -th secant variety of the d -th Veronese embedding of $\mathbb{P}W$. If $d \geq 2r$, $r \leq i \leq d - r$ and $r \leq 12$ then $\sigma_r(v_d(\mathbb{P}W))$ is set-theoretically defined by $(r + 1) \times (r + 1)$ minors of the i -th catalecticant matrix.*

See [BuBu, §1.1] for explanation of notation used in the statement and [BuBu, Thm 1.6, Cor. 1.11] for the resulting of this theorem from Theorem 2. In this language Theorem 1 may be restated as: any form of degree $d \geq 3$ in $V \oplus \mathbb{C}z$, which can be written as $f \cdot z^{d-3}$ where $f \in S^3V$, belongs to the 12-th secant variety of d -th Veronese embedding of $\mathbb{P}(V \oplus \mathbb{C}z)$ (see [BuBu, §8.1]).

The idea of the proof of Theorem 1 is to choose an appropriate apolar algebra $A = S^\bullet V^*/I_F$ and prove that it has Hilbert function $(1, 5, 5, 1)$, it is smoothable and the Zariski tangent space at the point $[\text{Spec } A] \in \text{Hilb}_{12} \mathbb{P}W$ has dimension $60 = 12 \cdot 5$.

Lemma 4. *The ideal I_F apolar to the form*

$$F := x_2^2 \cdot x_5 + x_2 \cdot x_4^2 + x_1^2 \cdot x_5 + x_3^2 \cdot x_4 \in S^3V \quad (1)$$

is generated by

$$a_1 \cdot a_2, a_1 \cdot a_3, a_1 \cdot a_4, a_2^2 - a_1^2, a_2 \cdot a_3, a_2 \cdot a_4 - a_3^2, a_2 \cdot a_5 - a_4^2, a_3 \cdot a_5, a_4 \cdot a_5, a_5^2$$

and its residue algebra A has the Hilbert function $h_A = (1, 5, 5, 1)$.

Proof. By computing the partial derivatives of F we conclude that no linear form annihilates F , thus $h_A(1) = 5$. The Hilbert function h of a local f.d. graded Gorenstein algebra having socle degree d satisfies $h(d - n) = h(n)$ where $0 \leq n \leq d$, thus $h_A = (1, 5, 5, 1)$.

A straightforward check shows that the given elements annihilate F . Denote by I the ideal generated by these elements. A direct check or a computation (see source `quotientDimension`) shows that the residue algebra of I has dimension 12 as a k -vector space so I is the apolar ideal of F . \square

Lemma 5. *Let A_0 be a finitely dimensional k -algebra and $A := A_0[x]/Jx$, where $J \triangleleft A_0$, be a graded algebra.*

Let $f := m - x^2$, where $m \in J$, then there exists a flat family over $k[t]$ with special fiber isomorphic to A/f and a general fiber of the form $A_0[x]/I_1 \cap I_2$, where $I_1 + I_2 = 1$, $A_0[x]/I_1 \simeq A_0/mJ$ and $A_0[x]/I_2 \simeq A_0/J$.

Proof. Put $f_\alpha := m - \alpha x - x^2$ for any $\alpha \in k$. Consider $A[t]/(m - t \cdot x - x^2)$ over $k[t]$. The fiber over $t = 0$ is the algebra A/f . Take $\alpha \in k \setminus \{0\}$. Since $(x^2, x + \alpha) = (1)$ and $x(m - \alpha x - x^2) = -\alpha x^2 - x^3$ the following equality of A -ideals holds

$$(f_\alpha, x^2) \cap (f_\alpha, x + \alpha) = (f_\alpha, x^2) \cdot (f_\alpha, x + \alpha) = (f_\alpha, x^3 + \alpha x^2) = (f_\alpha).$$

Now $(f_\alpha, x^2) = (m - \alpha x, x^2) = (m - \alpha x)$ so that $A/(f_\alpha, x^2) \simeq A_0/mJ$ and $(f_\alpha, x + \alpha) = (m, x + \alpha)$, so that $A/(f_\alpha, x + \alpha) \simeq A/(m, x + \alpha) \simeq A_0/J$.

Now $\dim_k A/f = \dim_k A_0/mJ + \dim_k A_0/J$, thus the length of fibers is constant. The algebra $A[t]/(m - tx - x^2)$ is finite over $k[t]$ so by [Har, Thm III.9.9] the family $k[t] \rightarrow A[t]/(m - tx - x^2)$ is flat, which proves the claim. \square

Corollary 6. *The apolar algebra of the form F defined in Lemma 1 is smoothable.*

Proof. Let $V' \subseteq V$ be spanned by a_2, a_3, a_4, a_5 .

It follows from Lemma 4 that the apolar ideal I_F of F has the form $J_0 + J_1 \cdot (a_1) + (a_2^2 - a_1^2)$ where $J_0, J_1 \triangleleft S^\bullet V'^*$, $J_0 \subseteq J_1 = (a_2, a_3, a_4)$, $a_2^2 \in J_1$ and $a_2^2 \cdot J_1 \subseteq J_0$.

We identify $S^\bullet V^*$ with the polynomial algebra $S^\bullet V'^*[a_1]$. Applying the previous lemma to the algebra $A_0 = S^\bullet V'^*/J_0$, ideal $J := J_1$ and element $m := a_2^2$ we see that the apolar algebra $S^\bullet V^*/I_F$ is a flat degeneration of the algebras of the form $S^\bullet V^*/I_1 \cap I_2$, where $I_1 + I_2 = 1$, $S^\bullet V^*/I_1 \simeq S^\bullet V'^*/J_0 + a_2^2 \cdot J_1 = S^\bullet V'^*/J_0$ and $S^\bullet V^*/I_2 \simeq S^\bullet V'^*/J_1$.

The quotient $S^\bullet V'^*/J_0$ is canonically isomorphic to the apolar algebra to the form $F(x_1 = 0) = x_2^2 \cdot x_5 + x_2 \cdot x_4^2 + x_3^2 \cdot x_4$. One can check, as in proof of Lemma 1, that the Hilbert function of this algebra is $(1, 4, 4, 1)$ so it is Gorenstein and thus smoothable (see [CN10]). Now a dimension count shows that $S^\bullet V^*/I_2$ is a 2-dimensional algebra over k , so it is smoothable as well.

Finally smoothability can be checked on irreducible components ([CEVV8ar, §4.1]), so the smoothability of both $S^\bullet V^*/I_1$ and $S^\bullet V^*/I_2$ implies the smoothability of $S^\bullet V^*/I_1 \cap I_2$ proving that $S^\bullet V^*/I_F$ is a flat degeneration of smoothable algebras and thus it is smoothable. \square

Lemma 7. *The tangent space to the Hilbert scheme of V^* at point $[R]$ corresponding to the scheme R , where I_F is the apolar ideal to F defined in 1 and $R := \text{Spec } S^\bullet V^*/I_F$, has dimension 60.*

Proof. The inclusion $V^* \subseteq \mathbb{P}W$ induces an inclusion of Hilbert schemes. The tangent space to a finite projective scheme defined by the saturated homogeneous ideal I in the total coordinate ring $S = k[W]$ is canonically identified with $\dim_k \text{Hom}_S(I, S/I)$ (see [Har2, Thm 1.1b]).

View R as the projective scheme defined by the homogenization of the ideal I , which is easily seen to be saturated. We can identify $\text{Hom}_S(I, S/I) = \text{Hom}_S(I/I^2, S/I) = \text{Hom}_{S/I}(I/I^2, S/I)$. Now S/I is Gorenstein, thus self injective, so $\text{Hom}(-, S/I)$ is exact and by looking on small extensions we see that $\dim_k \text{Hom}_{S/I}(I/I^2, S/I) = \dim_k I/I^2 = \dim_k S/I^2 - \dim_k S/I$. A computer algebra check shows that this dimension is equal to 60 (see source `quotientDimension`). \square

Proof of Theorem 1

Proof. Recall that we have fixed an inclusion $V^* \subseteq \mathbb{P}W$. A local f.d. Gorenstein algebra having Hilbert function $(1, 5, 5, 1)$ is isomorphic to the algebra of global sections of a finite closed Gorenstein subscheme of $\mathbb{P}W$. The locus \mathcal{G} of points corresponding to such subschemes is an irreducible subset of $\text{Hilb}_{12} \mathbb{P}W$ by [Ia, Thm I], it suffices to prove that it is contained in $\text{Hilb}_{12}^\circ \mathbb{P}W$.

Suppose on contrary that the locus \mathcal{G} is contained in an irreducible component of $\text{Hilb}_{12} \mathbb{P}W$ other than the closure of smooth schemes. The scheme R defined in Lemma 7 is smoothable by Cor. 6, so $[R]$ lies in the intersection of these two components, thus the dimension of the Zariski tangent space at $[R]$ is greater than the dimension of $\text{Hilb}_{12} \mathbb{P}W$, which is 60. This contradicts Lemma 7. \square

2 Magma source code

The following [Magma] source code may be helpful in proving Lemmas 1 and 7. In fact earlier versions of the proof used much more of the Magma computational capacity. Magma console is available for use on-line at <http://magma.maths.usyd.edu.au/calc/>.

```

quotientDimension := function(ideal_)
    return Dimension(quo<Generic(ideal_) | ideal_>);
end function;

Q := RationalField();
P<a1, a2, a3, a4, a5> := PolynomialRing(Q, 5);
I := Ideal([a1*a2, a1*a3, a1*a4, a2^2 - a1^2, a2*a3,
            a2*a4 - a3^2, a2*a5 - a4^2, a3*a5, a4*a5, a5^2]);

quotientDimension(I);
quotientDimension(I^2) - quotientDimension(I);

```

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